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The response of a quantum field to classical chaos

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Abstract. The quantum theory of a massless scalar field in a box with one chaotically moving wall is presented. The time dependence of normally ordered coherence functions is found to be chaotic, whereas the number of quanta present is insensitive to the details of the chaotic motion.

1. Introduction

The study of quantum chaos [1] is attracting attention although there is no definition of it which is accepted universally. The studies to date have been concerned with systems with effectively a small number of degrees of freedom. Chaos in quantum field theory has not been discussed. Such a discussion is clearly going to be a difficult one. Classical chaos in the early days was analysed for idealised models. One such system [2] involves a massive Newtonian particle bouncing in a container with a moving wall. The particle motion showed stochasticity. A natural generalisation of this to an infinite number of degrees of freedom is the dynamics of a massless scalar quantum field which vanishes outside a box with one chaotically oscillating wall. The study of such a model system, we hope, will be a useful preliminary step in the understanding of the interplay of chaos and quantum fields. The specialisation to a scalar field is not unnecessarily restrictive, since, if we were to consider a polarised electromagnetic field in the box, the polarisation of the field would be unaffected by the type of motions of interest to us. The magnitude of effects in the electromagnetic case will be small unless the motion of the wall is relativistic which is difficult to achieve. However, the same model could in principle apply to an elastic medium (where instead of photons we deal with phonons) and this objection is less important. At this stage the model should be regarded purely as a theoretical laboratory.

The method of quantisation that we will adopt is already known [3, 4]. It is essentially canonical quantisation which is adapted to quantum field theories with moving boundaries. Within the subject of chaology it seems to be little known. Since it is within the context of chaos that we wish to present our considerations we will outline the method. The accelerating wall excites the zero-point energy in the box. In our case the wall motion is chaotic and the accelerations result from this motion. Can we distinguish in the quantum field theory the chaotic nature of the motion from other simpler motion? The answer is that the correlation (or coherence) functions for the field operators can distinguish between chaos and non-chaos. In contrast the spectrum of excitations for the motion cannot. For clarity we will initially restrict our calculations to slow wall motions and also for simplicity to a one-space and one-time field theory.

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The qualitative features of our results are unchanged if restriction to slow wall motions is relaxed, and we discuss the reasons for this at the end of the next section.

2. Quantisation in the presence of moving boundaries

We will consider the method of canonical quantisation. For a finite-dimensional theory with a 2N-dimensional phase space it is necessary to choose coordinates $\{q^i\}_{1\leq i\leq N}$ and conjugate momenta $\{p_i\}_{1\leq i\leq N}$. These operators are then required to satisfy the commutation

$$[q^l, p_m] = \mathrm{i}\hbar\delta^l_m. \tag{1}$$

It is helpful to consider this very familiar procedure from a more abstract viewpoint. If Q is the classical space of the coordinate $\boldsymbol{q}$, then the classical phase space $(\boldsymbol{q}, \boldsymbol{p})$ forms a cotangent bundle [5] T^*Q. This bundle is a finite-dimensional symplectic manifold [5] since it has a natural non-degenerate 2-form σ

$$\sigma = \sum_{i=1}^{N} \mathrm{d}q^i \wedge \mathrm{d}p_i \tag{2}$$

whose exterior derivative is zero. The Poisson bracket $\{f, g\}$ of two smooth functions f and g, from T^*Q to the reals, can be defined in terms of σ:

$$\{f, g\} = \sigma(\xi_f, \xi_g) \tag{3}$$

where

$$\xi_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{4}$$

Although there are many subtleties in the general quantisation programme, a standard approach is to obtain the canonical commutation relations by replacing the Poisson brackets for a special set of classical observables by commutators. We are interested in the field theory of a massless scalar field and, akin to all field theories, it is infinite dimensional. A symplectic form analogous to σ needs to be found. The classical field equation is, of course,

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0 \tag{5}$$

and $\varphi(t, x) \in H^1(\mathbb{R})$, $\partial\varphi(t, x)/\partial t \in L^2(\mathbb{R})$. The wavevelocity has been taken to be one. Moreover, $L^2(\mathbb{R})$ is the space of measurable functions defined almost everywhere on $\mathbb{R}$ such that $|f|^2$ is integrable, and $H^1(\mathbb{R})$ is the space of functions with first derivatives in $L^2(\mathbb{R})$. For T^*Q we will take $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and the corresponding σ

$$\sigma((h_0, l_0), (h_1, l_1)) = \int (l_0 h_1 - l_1 h_0)\, \mathrm{d}x \tag{6}$$

where (h_0, l_0) and $(h_1, l_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. This symplectic structure of the space of classical solutions [5] can be quantised as in the finite-dimensional case by associating operators with a 'canonical' basis of T^*Q such that the commutators are given by the 2-form σ on the corresponding canonical classical solutions. We will give the detailed correspondence after having introduced the modification for moving boundaries. If

$x = b_1(t)$ is the left-most boundary and $x = b_2(t)$ is the other boundary, then equation (3) is for fields $\varphi(x, t) \in H^1(I)$ and $\partial\varphi(t, x)/\partial t \in L^2(I)$ where I is the interval $b_1(t) \leqslant x \leqslant b_2(t)$. Hence for two solutions $\varphi_1(x, t)$ and $\varphi_2(x, t)$ the action of σ gives

$$\sigma(\varphi_1, \varphi_2) = (\varphi_1|\varphi_2) = \int_{b_1(t)}^{b_2(t)} \varphi_2(x, t) \frac{\overleftrightarrow{\partial}}{\partial t} \varphi_1(x, t)\, \mathrm{d}x$$

which we will simply denote by $(\varphi_1|\varphi_2)$. From the field equations and the boundary conditions (equivalent to a tangential component of the electric field vanishing at the wall or mirror)

$$\varphi(b_1(t), t) = \varphi(b_2(t), t) = 0 \tag{7}$$

it is easy to see that $(\varphi_2|\varphi_1)$ is independent of time. It is possible to choose two sets of classical solutions $\{u_m(x, t)|m = 1, \ldots, \infty\}$ and $\{v_m(x, t)|m = 1, \ldots, \infty\}$ such that [5]

$$(u_m|u_n) = 0 \qquad (v_m|v_n) = 0 \qquad (u_m|v_n) = \delta_{mn}. \tag{8}$$

In the appendix an explicit construction for such sets is given in terms of a complete set of real orthonormal functions on the interval $b_1(0) \leqslant x \leqslant b_2(0)$. The theory is quantised by associating operators p_n with u_n and q_n with v_n, and the commutator $[p_n, q_m]$ is defined to be

$$[p_n, q_m] = -\mathrm{i}(u_n|v_m) = -\mathrm{i}\delta_{nm} \tag{9}$$

(units have been chosen so that $\hbar = 1$).

The other commutators are defined similarly. We will restrict ourselves to $b_1(t) = b_1(0) = 0$ for all t, i.e. the left-most mirror is stationary. It is possible to show [4] that

$$\begin{aligned} u_n(x, t) &= (2n\pi)^{-1/2}[\cos(n\pi R(t-x)) - \cos(n\pi R(t+x))] \\ v_n(x, t) &= (2n\pi)^{-1/2}[\sin(n\pi R(t+x)) - \sin(n\pi R(t-x))] \end{aligned} \tag{10}$$

where R is twice differentiable, invertible and satisfies

$$R(t - b_2(t)) = R(t + b_2(t)) - 2 \tag{11}$$

(we will, for convenience, describe this in the appendix).

So far we have been rather general. It is necessary now to have some idea of the solution of (11) at the analytic level and find what properties of the chaotic motion are reflected in $R(t)$. This is helped by noticing that for a mirror moving with velocity v (i.e. $b_2(t) = vt$)

$$R(t) = (\tanh^{-1} v)^{-1} \log t$$

which for small v reduces to

$$R(t) \sim \frac{1}{v} \log t = \int^t \frac{\mathrm{d}t'}{vt'} = \int^t \frac{\mathrm{d}t'}{b_2(t')} \tag{12}$$

(the last integral relation for $R(t)$ is valid also for a stationary wall).

If we are to obtain $R(t)$ in the general case it seems difficult to make progress without restricting attention to $|\dot{b}_2(t)| \ll 1$. In common with studies in chaos we will be interested in the behaviour of the quantum field at large times, i.e. for times such that the ratio of $b_2(t)$ to the unperturbed length of the cavity is much larger than

$1+b_2(t)$. This allows a systematic solution of (11) and it is possible to write (motivated by (12))

$$R(t\pm x)=\int^t \frac{\mathrm{d}t'}{b_2(t')}+g^{(\pm)}\left(\frac{x}{b_2(t)}, \varepsilon t\right). \tag{13}$$

$g^{(\pm)}$ is a function that needs to be determined and ε is the order of $|\dot{b}_2(t)|$. The indefiniteness of the integrals is not of concern since R is indeterminate up to a constant. In fact if $R\to R+\alpha$ then

$$\begin{aligned} u_n &\to u_n' = \cos(n\pi\alpha)u_n+\sin(n\pi\alpha)v_n \\ v_n &\to v_n' = \cos(n\pi\alpha)v_n-\sin(n\pi\alpha)u_n \end{aligned} \tag{14}$$

which is just an orthogonal transformation and the commutation relations are unaffected. Since ε is small it is possible to write [4]

$$g^{(\pm)}(\xi, \varepsilon s)=\sum_{n=0}^{\infty} g_n^{(\pm)}(\xi, s)\varepsilon^n. \tag{15}$$

We can show that

$$\begin{aligned} g^{(+)}(\xi, \varepsilon t) &= \xi+\varepsilon\left[\frac{1}{6}\int^{\varepsilon t} \mathrm{d}s\left(-2\frac{(\bar{b}_2'(s))^2}{\bar{b}_2(s)}+\bar{b}_2''(s)\right)-\tfrac{1}{2}\xi^2\bar{b}_2'(t)\right] \\ &\quad +\tfrac{1}{6}\varepsilon^2(\xi-\xi^3)[-2(\bar{b}_2'(t))^2+\bar{b}_2(t)\bar{b}_2''(t)]+\mathrm{O}(\varepsilon^3) \end{aligned} \tag{16}$$

where $\bar{b}_2(\varepsilon t)\equiv b_2(t)$, i.e. $\mathrm{d}/\mathrm{d}s\ \bar{b}_2(s)\ (=\bar{b}_2'(s))$ is of order 1.

A similar expression holds for $g^{(-)}(\xi, \varepsilon t)$. (The details of the equation satisfied by $g^{(\pm)}(\xi, \varepsilon t)$ will be given in the appendix.) We will now utilise these results to discuss what signatures a chaotic wall motion will leave on the quantum field theory. The type of motion that we will consider has the prescribed form

$$b_2(t)=\begin{cases} L & \text{for } t<0 \\ b(t) & \text{for } t_2>t\geqslant 0 \\ b(t_0) & \text{for } t\geqslant t_0 \end{cases} \tag{17}$$

where

$$b(0)=L \qquad \dot{b}(0)=0 \qquad \dot{b}(t_0)=0.$$

We will take advantage of the freedom given in (14) by choosing the lower limit of the integrals in (16) to be t_0. In our case $b(t)$ will be taken to be chaotic. Generally we can write

$$b(t)=\int_{-\infty}^{\infty} f(\omega)\exp(\mathrm{i}\omega t)\,\mathrm{d}\omega \tag{18}$$

with

$$f(-\omega)=f^*(\omega).$$

By chaotic motion of the wall we will mean that $f(\omega)$ is broadband. When the mirror is stationary for $t<0$ it is easy to describe the state of the field in terms of harmonic oscillator Fock states. Given an operator set $\{p_m, q_m|m=1,\ldots,\infty\}$ and a vacuum state $|0\rangle$ such that

$$\frac{1}{\sqrt{2}}(q_m+\mathrm{i}p_m)|0\rangle=0 \tag{19}$$

the general states are spanned by a basis $|\boldsymbol{n}\rangle$ defined as

$$|\boldsymbol{n}\rangle = \left(\frac{1}{\sqrt{2}}(q_1 - \mathrm{i}p_1)\right)^{n_1} \dots \left(\frac{1}{\sqrt{2}}(q_m - \mathrm{i}p_m)\right)^{n_m} \dots |0\rangle \tag{20}$$

and $\boldsymbol{n} = (n_1, \dots, n_m, \dots)$. An essential type of quantity that we need to calculate is the expectation value of the field operator at time $0 < t < t_0$ in the states $|\boldsymbol{n}\rangle$. We can say that we have the join of a stationary 'motion' with the chaotic motion at $t = 0$, and we need to find the relationship of the field for $t > 0$ to that for $t < 0$. In general if there is join of two motions of the wall $b^{(1)}(t)$ and $b^{(2)}(t)$ at $t = t'$ then the quantum field can be written as

$$\begin{aligned} \varphi(x, t) &= \sum_n (v_n^{(1)}(x, t)p_n^{(1)} - u_n^{(1)}(x, t)q_n^{(1)}) && \text{for } t < t' \\ &= \sum_n (v_n^{(2)}(x, t)p_n^{(2)} - u_n^{(2)}(x, t)q_n^{(2)}) && \text{for } t > t'. \end{aligned} \tag{21}$$

On referring to the appendix it will be clear that for $i = 1, 2$

$$\begin{aligned} u_n^{(i)}(x, t) &= (2n\pi)^{-1/2}\{\cos[n\pi R^{(i)}(t - x)] - \cos[n\pi R^{(i)}(t + x)]\} \\ v_n^{(i)}(x, t) &= (2n\pi)^{-1/2}\{\sin[n\pi R^{(i)}(t + x)] - \sin[n\pi R^{(i)}(t - x)]\} \end{aligned} \tag{22}$$

and

$$R^{(i)}(t - b^{(i)}(t)) - R^{(i)}(t + b^{(i)}(t)) + 2 = 0.$$

It is necessary to obtain the relation between the operators $p_n^{(1)}$, $q_n^{(1)}$ and $p_n^{(2)}$, $q_n^{(2)}$. With the help of such a relation it is possible to calculate the expectation value of the field operators for $t > t'$ in the Fock space defined by $p_n^{(1)}$, $q_n^{(1)}$. In a similar way it is possible to calculate the expectation value of field operators for $t < t'$ in the Fock space defined by $p_n^{(2)}$, $q_n^{(2)}$. An example of such a relation is

$$p_m^{(1)} = (\varphi | u_m^{(1)}) = \sum_n [(v_n^{(2)} | u_m^{(1)})p_n^2 - (u_n^{(2)} | u_m^{(1)})q_n^{(2)}]. \tag{23}$$

We must note a somewhat technical point here. In the definition of $(a|b)$ it is necessary that a and b are defined over the same interval. For $a = v_m^{(1)}$ and $b = u_n^{(2)}$ the intervals $0 \leqslant x \leqslant b^{(1)}(t)$ and $0 \leqslant x \leqslant b^{(2)}(t)$ do not coincide. In order to have a consistent definition away from $t = t'$ we need to define the propagation function $D(x, t; x', t')$. It solves the problem of the evolution of a classical configuration space function $f(x, t')$ for $t > t'$ since

$$f(x, t) = \int_{b_1(t')}^{b_2(t')} D(x, t; x't') \frac{\overleftrightarrow{\partial}}{\partial t'} f(x', t')\, \mathrm{d}x' \tag{24}$$

where b_1 and b_2 are as in (7). (We hope that there is no confusion with $b^{(1)}$ and $b^{(2)}$ introduced before (21).) Further details of D are given in the appendix. With the help of D we can write

$$v_n^{(1)}(x, t) = -(D(x, t; ., t') | v_n^{(1)}(., t')) \qquad \text{for } t > t'.$$

Continuing now with the relations of the form given in (23) it is easy to show that

$$p_n^{(2)} = \sum_m [(v_m^{(1)} | u_n^{(2)})p_m^{(1)} - (u_m^{(1)} | u_n^{(2)})q_m^{(1)}] \tag{25}$$

and

$$q_n^{(2)} = \sum_m [(v_m^{(1)} | v_n^{(2)})p_m^{(1)} - (u_m^{(1)} | v_n^{(2)})q_m^{(1)}]. \tag{26}$$

The conclusions summarised in (19)-(26) are quite general. We will evaluate the consequences for the motion of (17). For non-trivial features it is necessary to work to $O(\varepsilon^2)$. Later we will argue quite generally that our conclusions are unaffected by higher-order terms in ε. In this case

$$b^{(1)}(t) = L$$

and so

$$R^{(1)+}(x, t) = (x+t)/L. \tag{27}$$

Hence

$$u_n^{(1)}(x, t) = \frac{1}{(2n\pi)^{1/2}}\left[\cos\left(\frac{n\pi(t-x)}{L}\right) - \cos\left(\frac{n\pi(t+x)}{L}\right)\right] \tag{28}$$

$$v_n^{(1)}(x, t) = \frac{1}{(2n\pi)^{1/2}}\left[\sin\left(\frac{n\pi(t+x)}{L}\right) - \sin\left(\frac{n\pi(t-x)}{L}\right)\right].$$

Similarly

$$R^{(2)+}(x, t_0) = \frac{x}{L} + \frac{1}{6}\left(\frac{x}{L} - \frac{x^3}{L^3}\right)(L\ddot{b}(t_0)) + O(\varepsilon^3). \tag{29}$$

It is straightforward to show that

$$\begin{aligned}(v_n^{(1)}|v_n^{(2)}) &= 0 + O(\varepsilon^3)\\ (u_k^{(1)}|u_n^{(2)}) &= 0 + O(\varepsilon^3)\end{aligned} \tag{30}$$

and

$$(v_k^{(1)}|u_n^{(2)}) = \left(\frac{k}{n}\right)^{1/2}\left(\delta_{kn} - (-1)^{k-n}(1-\delta_{kn})\frac{nL\ddot{b}(0)}{\pi^2(k-n)^3} + (-1)^{k+n}\frac{nL\ddot{b}(0)}{\pi^2(k+n)^3}\right) + O(\varepsilon^3) \tag{31}$$

and

$$(u_k^{(1)}|v_n^{(2)}) = -\left(\frac{k}{n}\right)^{1/2}\left(\delta_{kn} - (-1)^{k+n}\frac{nL\ddot{b}(0)}{\pi^2(k-n)^3} - (-1)^{k-n}(1-\delta_{kn})\frac{nL\ddot{b}(0)}{\pi^2(k-n)^3}\right) + O(\varepsilon^3). \tag{32}$$

Hence from (25) and (26) we can deduce that

$$q_n^{(2)} = q_n^{(1)}\left(1 - \frac{L\ddot{b}(0)}{8\pi^2 n^2}\right) - \sum_{k(\neq n)} (-1)^{k+n}\frac{(kn)^{1/2}L\ddot{b}(0)}{\pi^2}\left(\frac{1}{(k+n)^3} + \frac{1}{(k-n)^3}\right)q_k^{(1)} + O(\varepsilon^3) \tag{33}$$

and

$$p_n^{(2)} = p_n^{(1)}\left(1 + \frac{L\ddot{b}(0)}{8\pi^2 n^2}\right) + \sum_{k(\neq n)} (-1)^{k+n}\frac{(kn)^{1/2}L\ddot{b}(0)}{\pi^2}\left(\frac{1}{(k+n)^3} - \frac{1}{(k-n)^3}\right)p_k^{(1)} + O(\varepsilon^3). \tag{34}$$

Clearly $q_n^{(2)}$ and $p_n^{(2)}$ do not depend on the detailed nature of the chaotic motion of the wall. As a result, given an initial stage $|\boldsymbol{n}^{(1)}\rangle$ (similar to $|\boldsymbol{n}\rangle$ of (20) except that $(q_m^{(1)}, p_m^{(1)})$ replaces (q_m, p_m) for all m) expectation values of powers of $p_n^{(2)}$ and $q_n^{(2)}$ are of no avail in giving indicators about the chaotic motion. Equivalently the expectations of the number of different excitations produced by the chaotically moving wall are quite insensitive to the details of the chaotic motion.

The content of a field theory is best seen in the behaviour of normally ordered coherence functions. It is necessary to determine whether these functions are sensitive to the chaotic motion. This is easily done. The coherence functions are defined to be

$$\langle|\varphi^{(-)}(x_1, t_1)\ldots\varphi^{(-)}(x_r, t_r)\varphi^{(+)}(x_1, t_1)\ldots\varphi^{(+)}(x_s, t_s)|\rangle$$

where $|\ \rangle$ is an initial state of the form $|\boldsymbol{n}^{(1)}\rangle$.

Here

$$\varphi^{(-)}(x, t) = -\frac{1}{\sqrt{2}}\sum_n (u_n^{(2)}(x, t) - \mathrm{i}v_n^{(2)}(x, t))a_n^{(2)\dagger} \tag{35}$$

$$(\varphi^{(-)}(x, t))^{\dagger} = \varphi^{(+)}(x, t)$$

and

$$a_n^{(2)} = \frac{1}{\sqrt{2}}(q_n^{(2)} + \mathrm{i}p_n^{(2)}).$$

On substituting the expressions in (22) and (13) into (35) we see that the coherence function depends explicitly on $b^{(2)}(t)$ and not just on the time integral of $b^{(2)}(t)$. The exact form of the dependence requires us to specify the initial state $|\ \rangle$. For a particularly simple initial state $|\ \rangle$ with a single excitation in the mode n_0 we have

$$|\ \rangle = a_{n_0}^{(1)\dagger}|0\rangle$$

and the second-order coherence function $\langle|\varphi^{(-)}(x, t)\varphi^{(+)}(x, t)|\rangle$ is

$$\frac{1}{4\pi}\Bigg|\frac{1}{n_0^{1/2}}[\exp(-\mathrm{i}n_0\pi R^{(2)-}(x, t)) - \exp(-\mathrm{i}n_0\pi R^{(2)+}(x, t))]$$
$$-\frac{(-1)^{n_0}}{\pi^2}n_0^{1/2}L\ddot{b}(0)\sum_{n(\neq n_0)}(-1)^n\frac{1}{(n_0-n)^3}$$
$$\times[\exp(-\mathrm{i}n\pi R^{(2)-}(x, t)) - \exp(-\mathrm{i}n\pi R^{(2)+}(x, t))]\Bigg|^2.$$

It is now necessary to extract the time dependence of this coherence function. We can write (cf (13) and (16))

$$R^{(\pm)}(x, t) = a(x, t) + c^{(\pm)}(x, t) + \mathrm{O}(\varepsilon^3)$$

where

$$a(x, t) = \int_{t_0}^{t}\frac{\mathrm{d}t'}{b(t')} + \frac{1}{6}\int_{t_0}^{t}\mathrm{d}t'\left(-\frac{2(\dot{b}(t'))^2}{b(t')} + \ddot{b}(t')\right) - \frac{1}{2}\frac{x^2}{b^2(t)}\dot{b}(t)$$

$$c^{(\pm)}(x, t) = \pm\left[\frac{x}{b(t)} + \frac{1}{6}\left(\frac{x}{b(t)} - \frac{x^3}{(b(t))^3}\right)\left(-2(\dot{b}(t))^2 + b(t)\ddot{b}(t)\right)\right].$$

Moreover, both a and $c^{\pm}$ are real. It is then easy to show that

$$\langle|\varphi^{(-)}(x,t)\varphi^{(+)}(x,t)|\rangle=\frac{1}{\pi}\left|\frac{-1}{n_0^{1/2}}\sin(n_0\pi c^-(x,t))\exp(-\mathrm{i}n_0\pi a(x,t))\right.$$
$$+(-1)^{n_0}\frac{n_0^{1/2}}{\pi^2}L\ddot{b}(0)\sum_{n(\neq n_0)}\frac{(-1)^n}{(n_0-n)^3}$$
$$\left.\times\exp(-\mathrm{i}n\pi a(x,t))\sin(n\pi c^-(x,t))\right|^2.$$

Since

$$\left|\sum_{n\neq n_0}\frac{(-1)^n}{(n_0-n)^3}\exp(-\mathrm{i}n\pi a(x,t))\sin(n\pi c^-(x,t))\right|\leqslant\sum_{n\neq n_0}\frac{1}{(n_0-n)^3}<\infty$$

and the series

$$\sum_{n\neq n_0}\frac{(-1)^n}{(n_0-n)^3}$$

is absolutely convergent uniformly in x and t, given a prescribed accuracy δ there is a N_δ such that

$$\sum_{\substack{n\leqslant N_\delta\\(n\neq n_0)}}\frac{(-1)^n}{(n_0-n)^3}\exp(-\mathrm{i}n\pi a(x,t))\sin(n\pi c^-(x,t))$$

approximates the infinite series to within δ for all x and t. With further analysis we can show that the condition

$$N_\delta>\left(\frac{1}{\delta}+n_0\right)$$

is sufficient to guarantee that the truncated series is accurate to within δ. It is clear that $a(x,t)$ and $c^{(\pm)}(x,t)$ are chaotic since they are expressed directly in terms of $b(t)$. The expression

$$\frac{1}{\pi}\left|-\frac{1}{n_0^{1/2}}\sin(n_0\pi c^-(x,t))\exp(-\mathrm{i}n_0\pi a(x,t))+(-1)^{n_0}\frac{n_0^{1/2}}{\pi^2}L\ddot{b}(0)\right.$$
$$\left.\times\sum_{\substack{n\leqslant N_\delta\\(n\neq n_0)}}\frac{(-1)^n}{(n_0-n)^3}\exp(-\mathrm{i}n\pi a(x,t))\sin(n\pi c^-(x,t))\right|^2$$

is chaotic, as it is a finite sum where all the individual terms are chaotic. As a result $\langle|\varphi^{(-)}(x,t)\varphi^{(+)}(x,t)|\rangle$ is to any arbitrary accuracy well represented by a chaotic time dependence for all time.

This low-order calculation is illustrative of the situation in general. In fact the quantities

$$(v_n^{(1)}|u_m^{(2)})\qquad(u_n^{(2)}|u_m^{(1)})\qquad(v_n^{(2)}|v_m^{(1)})\qquad(u_n^{(2)}|v_m^{(1)})$$

are insensitive to the details of chaotic motion. Indeed we have relations such as

$$(u_m^{(1)}|v_n^{(2)})=\frac{1}{2\pi(mn)^{1/2}}\int_{-b^{(2)}(t')}^{b^{(2)}(t')}\left(\cos(m\pi R^{(1)}(t'+x))\frac{\overleftrightarrow{\partial}}{\partial x}\sin(n\pi R^{(2)}(t'+x))\right)$$

and the integration smears out any dependence on the form of chaotic motion of the wall. Hence the $p_n^{(i)}$ and $q_n^{(i)}$ which are determined by these entities will not be sensitive to the chaos. However, from (16) we see that the functions $R^{(i)}$ are sensitive. Consequently, the field $\varphi(x, t)$ will reflect the chaotic motion during the time the mirror is moving chaotically. The resulting coherence functions will thus have a chaotic time dependence.

We have shown the interplay between chaos and a quantum field for a simple model. An interesting next step would be to introduce intrinsic non-linearity of the dynamics of the quantum field. From our analysis we could speculate that a more complicated model with self-generated chaotic structure in the cavity modes would have coherence functions which show chaos.

Appendix

We will prove some of the results referred to in the text. First we will give arguments to support the existence of functions satisfying (8). For the interval $b_1(0) \leq x \leq b_2(0)$ it is possible to construct a complete set of orthonormal functions which vanish at the endpoints. Such a set is

$$\left\{ f_n(x) | f_n(x) = \left(\frac{2}{b_2(0)-b_1(0)}\right)^{1/2} \sin\left(\frac{n\pi(x-b_1(0))}{b_2(0)-b_1(0)}\right) \begin{array}{l} \text{for } b_1(0) \leq x \leq b_2(0) \\ \text{and } b_2(0) > b_1(0) \end{array} \right\}. \tag{A1}$$

The quantities u_n and v_m are constructed as

$$u_n(x, t) = \int_{b_1(0)}^{b_2(0)} D(x, t; x', 0) f_n(x') \, \mathrm{d}x' \tag{A2}$$

$$v_n(x, t) = -\int_{b_1(0)}^{b_2(0)} \frac{\partial}{\partial t'} D(x, t; x', t') \bigg|_{t'=0} f_n(x') \, \mathrm{d}x'. \tag{A3}$$

For this to be a correct solution it is necessary that $D(x, t; x', t')$ satisfies the wave equation in both (x, t) and (x', t'), together with the boundary conditions of (7). Without this u_n and v_n would not be solutions of the wave equation with correct boundary condition. In order to have

$$(u_n | v_m) = \delta_{nm}$$

we find it necessary to require that

$$\frac{\partial}{\partial t} D(x, t; x', t') \bigg|_{t=t'} = -\frac{\partial}{\partial t'} D(x, t; x't') \bigg|_{t=t'} = \delta(x - x'). \tag{A4}$$

Other conditions on D can be found in a similar way. We have converted the problem of finding $u_n(x, t)$ and $v_m(x, t)$ into the one of constructing $D(x, t; x', t')$. Such a D can be constructed but we will not give it here.

There is another more direct way of obtaining a set of classical solutions vanishing at the boundaries. For simplicity we will consider $b_1(t) = 0$. Instead of (t, x) we transform to the pair of variables (u, s)

$$R(t-x) = u - s \qquad R(t+x) = u + s \tag{A5}$$

where R is an invertible function and twice differentiable. We also require that $s=0$ and $s=1$ correspond to the boundaries $x=0$ and $x=b_2(t)$. It is easy to deduce that

$$R(t+b_2(t))=u+1 \qquad R(t-b_2(t))=u-1$$

and so [4]

$$R(t+b_2(t))-R(t-b_2(t))=2. \tag{A6}$$

Moreover, the equation for φ in terms of u and s is again a wave equation. The problem is reduced to finding the modes of the classical wave equation with fixed boundaries. A complete set of solutions is given by

$$u_n=(2/\pi n)^{1/2}\sin \pi n u \sin \pi n s$$
$$v_n=(2/\pi n)^{1/2}\cos \pi n u \sin \pi n s$$

which can be rewritten

$$\begin{aligned} u_n&=(1/2\pi n)^{1/2}[\cos \pi n(u-s)-\cos \pi n(u+s)] \\ v_n&=(1/2\pi n)^{1/2}[\sin \pi n(u+s)-\sin \pi n(u-s)]. \end{aligned} \tag{A7}$$

These are equivalent to (10).

The solution of (A6) is in general not known. However, an analysis reminiscent of multi-time scale analysis in hydrodynamics is possible. If we write

$$R^{\pm}(x,t)=R(t\pm x) \tag{A8}$$

then

$$\frac{\partial}{\partial t}R^{\pm}(x,t)=\pm\frac{\partial R}{\partial x}$$

and, corresponding to (A6),

$$R^{\pm}(\mp b_2(t),t)=R^{\pm}(\pm b_2(t),t)-2. \tag{A9}$$

We introduce functions

$$g^{\pm}\left(\frac{x}{b_2(t)},\varepsilon t\right)=R^{\pm}(x,t)-\int^{t}\frac{1}{b_2(t')}\,\mathrm{d}t'. \tag{A10}$$

From (A8) we deduce that

$$(1\pm\xi\dot{b}_2(t))\frac{\partial}{\partial\xi}g^{\pm}(\xi,t)\mp b_2(t)\frac{\partial g^{\pm}}{\partial t}\mp 1=0. \tag{A11}$$

On assuming that $|\dot{b}_2(t)|\ll 1$ and g is slowly varying with time we can write the expansion of (15). The equations for $g^{(\pm)}$ obtained by equating powers of ε in (A11) can then be solved recursively in a straightforward manner.

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